

Ph.D. ENTRANCE EXAMINATION, APRIL 2022

STATISTICS

Time : Two Hours

Maximum : 100 Marks

Part A

Answer all questions.
Each question carries 2 marks.

- Let $X_1, X_2, \dots$ be independent and identically distributed random variables with $E(X_i) = 0$ and $\text{Var}(X_i) = 1$. Let $S_n = \sum_{i=1}^n X_i$ and $\Phi(x)$ be the cumulative distribution function of a standard normal distribution. Then for any $x > 0$ $\lim_{n \rightarrow \infty} P(-nx < S_n < nx)$ equals.
(a) $\Phi(x)$. (b) $1 - 2\Phi(x)$.
(c) $2\Phi(x) - 1$. (d) 1.
- Let A and B be $n \times n$ matrices such that $AB + A^2 = I - A^2B$, where I is an identity matrix. Which of the following is always true?
(a) A + B is singular. (b) B is singular.
(c) A is singular. (d) AB is singular.
- If $X_1, X_2, \dots, X_n$ is a random sample from a population with density function $f(x) = \frac{1}{2}e^{|x-\theta|}$, then based on the following data the MLE of the parameter θ is Data : 2, 4, 6, 2.5, 3.5, 3, 7.
(a) 4. (b) 3.5.
(c) 7. (d) 2.
- Let $X_1, X_2, \dots$ be independent and identically distributed normal random variables with mean 1 and variance 1. Let N be a Poisson random variable with mean 2, independent of $X_1, X_2, \dots$. Then the variance of $X_1 + X_2 + \dots + X_{N+1}$ is :
(a) 9. (b) 5.
(c) 3. (d) 4.

Turn over

5. Let $X_1, X_2, \dots, X_n$ be a random sample from Uniform $\left(\theta - \frac{1}{2}, \theta + \frac{1}{2}\right)$. Consider the problem of testing $H_0: \theta = -\frac{1}{2}$ against $H_1: \theta = \frac{1}{2}$. Define $X_{(1)} = \text{Min}\{X_1, X_2, \dots, X_n\}$. Consider the following test:

Reject H_0 if $X_{(1)} > 0$, accept otherwise. Which of the following is true?

- (a) Power of the test = 0, size of the test = 0.
 - (b) Power of the test = 0, size of the test = 1.
 - (c) Power of the test = 1, size of the test = 1.
 - (d) Power of the test = 1, size of the test = 0.
6. Suppose $D \sim N(0,1)$ and $U = \begin{cases} 1 & \text{if } D \geq 0 \\ 0 & \text{if } D < 0 \end{cases}$. Then the correlation coefficient between $|D|$ and U is equal to:
- (a) 1.
 - (b) 0.5.
 - (c) 0.
 - (d) 0.25.
7. Let C_x and C_y be coefficient of variations for the auxiliary variable and study variable and ρ is the correlation coefficient. Then ratio estimation is more efficient than sample mean when:

- (a) $\rho \frac{C_y}{C_x} > \frac{1}{2}$.
 - (b) $\rho \frac{C_y}{C_x} < \frac{1}{2}$.
 - (c) $\rho \frac{C_y}{C_x} < \frac{1}{2}$.
 - (d) $\rho \frac{C_x}{C_y} > \frac{1}{2}$.
8. $(X_n, n \geq 1)$ and X are random variables on a probability space. Suppose that X_n converges to X in probability. Which of the following is/are true?

(A) $E(|X_n - X|^2) \rightarrow 0$.

(B) $P(X_n \leq x) \rightarrow P(X \leq x)$ for all $x \in \mathbb{R}$.

(a) (A) only.

(b) (B) only.

(c) Both (A) and (B).

(d) Neither (A) nor (B).

9. Which of the following statements is/are true ?
- (A) Lebesgue measure of any line of finite length in R^k is zero.
(B) Lebesgue measure of any line of infinite length in R^k is zero.
- (a) (A) only. (b) (B) only.
(c) Both (A) and (B). (d) Neither (A) nor (B).
10. Let A be the likelihood ratio for testing the hypothesis $H_0 : \theta \in \Theta_0$ against $H_1 : \theta \in \Theta_1$. Then λ is a function of :
- (a) Every consistent estimator for θ .
(b) Every sufficient statistic for θ .
(c) Every unbiased estimator for θ .
(d) None of these.
11. Which of the following statements is/are true for a stochastic processes $\{X(t)\}$.
- (A) A strictly stationary processes with $\text{Var}(X(t)) < \infty$ is also covariance stationary.
(B) A covariance stationary processes with $X(t)$ follows normal distribution is strictly stationary.
- (a) (A) alone. (b) (B) alone.
(c) Both (A) and (B). (d) Neither (A) nor (B).
12. Assume that $X \sim \text{Binomial}(n, p)$ for some $n \geq 1$ and $0 < p < 1$ and $Y \sim \text{Poisson}(\lambda)$ for some $\lambda > 0$. Suppose $E[X] = E[Y]$. Then :
- (a) $\text{Var}(X) < \text{Var}(Y)$. (b) $\text{Var}(X) = \text{Var}(Y)$.
(c) $\text{Var}(X) > \text{Var}(Y)$. (d) None of the above.
13. Error degrees of freedom for a Graeco Latin square design of size 5 is :
- (a) 8. (b) 24.
(c) 4. (d) 12.

14. Let $X_1, X_2, \dots$ be independent and identically distributed standard normal random variables. Which of the following is true ?

- (a) $\frac{\sqrt{n} X_1}{\sqrt{X_1^2 + \dots + X_n^2}}$ has a t -distribution with n degrees of freedom.
- (b) $\frac{\sqrt{n} X_1}{\sqrt{X_2^2 + \dots + X_{n+1}^2}}$ has a t -distribution with n degrees of freedom.
- (c) $\frac{\sqrt{n} X_1}{\sqrt{X_2^2 + \dots + X_{n+1}^2}}$ has a t -distribution with $n - 1$ degrees of freedom.
- (d) $\frac{\sqrt{n} X_1}{\sqrt{X_1^2 + \dots + X_n^2}}$ has a t -distribution with $n - 1$ degrees of freedom.

15. Let $X_1, X_2, \dots, X_n$ be a random sample of size n from a p -variate normal distribution with mean μ and variance covariance matrix Σ . Choose the correct statement :

- (a) $(X_1 - \mu)' \Sigma^{-1} (X_1 - \mu)$ has Chi-square distribution with 1 d.f.
- (b) $X_1' X_1$ has Normal distribution.
- (c) $\sum_{i=1}^n (X_i - \mu) (X_i - \mu)'$ has Wishart distribution with n d.f.
- (d) $X_1 + X_2$ and $X_1 - X_2$ are independently distributed.

16. For an SPRT with strength (α, β) the boundary points A and B ($B < A$) satisfy.

- (a) $A \leq \frac{\beta}{1-\alpha}, B \geq \frac{1-\beta}{\alpha}$.
- (b) $A \leq \frac{1-\beta}{\alpha}, B \geq \frac{\beta}{1-\alpha}$.
- (c) $A \leq \frac{\alpha}{1-\beta}, B \geq \frac{\beta}{1-\alpha}$.
- (d) $A \leq \frac{\alpha}{1-\beta}, B \geq \frac{\beta}{1-\alpha}$.

17. Let T be an ancillary statistic for the parameter θ . Then T and S independent if :

- (a) S is sufficient for θ .
- (b) S is unbiased for θ .
- (c) S is complete sufficient for θ .
- (d) S is consistent for θ .

18. If f is continuous from R to R , then which of the following is/are correct ?

(A) $|f|$ is continuous.

(B) $\sqrt{f}$ is continuous.

(a) (A) alone.

(b) (B) alone.

(c) Both (A) and (B).

(d) Neither (A) nor (B).

19. A 2^4 factorial experiment with factors A, B, C and D is arranged in 4 blocks each of size 4. The arrangement is :

B_1	B_2	B_3	B_4
(1)	ab	a	b
abc	c	bc	ac
d	abd	ad	bd
$abcd$	cd	bcd	acd

Identified all confounded effects and interactions.

(a) AB, BC, AD.

(b) AB, BD, AD.

(c) AC, BD, ABCD.

(d) AB, BC, AC.

20. Let R be the sample correlation coefficient of a bivariate sample from a bivariate normal population

with correlation coefficient ρ . If $\rho = 0$ then the statistic $R\sqrt{\frac{n-2}{1-R^2}}$ follows a :

(a) t distribution with $(n-1)$ d.f.

(b) t distribution with $(n-2)$ d.f.

(c) F distribution with $(1, n)$ d.f.

(d) F distribution with $(1, n-1)$ d.f.

21. The distribution of Hotelling's T^2 statistic is :

(a) Chi-square distribution.

(b) t distribution.

(c) Wishart distribution.

(d) F distribution.

22. If $(X_1, X_2)'$ follows a bivariate normal distribution with dispersion matrix $\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$ then the

dispersion matrix of $(X_1 + X_2, X_1 - X_2)'$ is :

(a) $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$.

(b) $\begin{pmatrix} 3 & 1/2 \\ 1/2 & 1 \end{pmatrix}$.

(c) $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$.

(d) $\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$.

23. The stationary distribution of the following transition probability matrix is $\begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$.

(a) $\left(\frac{1}{3}, \frac{10}{27}, \frac{8}{27}\right)$.

(b) $\left(\frac{1}{3}, \frac{2}{5}, \frac{3}{5}\right)$.

(c) $\left(\frac{1}{3}, \frac{5}{9}, \frac{4}{27}\right)$.

(d) None of the above.

24. Consider the following statements on ratio and regression estimates of population mean $\bar{Y}$:

(A) The ratio estimator of $\bar{Y}$ is biased.

(B) The regression estimate is generally more efficient than the ratio estimator.

(a) Both (A) and (B).

(b) Neither (A) nor (B).

(c) (A) alone.

(d) (B) alone.

25. Let X be a random variable which follows a Poisson distribution with parameter λ and the prior distribution of λ follows $G(\alpha, \beta)$ then the posterior distribution of λ is:

(a) $G(\alpha+1, \beta+x)$.

(b) $G(\alpha+x, \beta)$.

(c) $G(\alpha+x, \beta+1)$.

(d) $G(\alpha, \beta+x)$.

(25 × 2 = 50 marks)

Part B

*Answer any five questions.
Each question carries 10 marks.*

26. Prove that every set of orthonormal vectors is linearly independent.
27. Obtain the UMP size α test for testing $H_0: \theta = \theta_0$ against $H_1: \theta \neq \theta_0$ in Uniform(0, θ).
28. Let $X_1, X_2, \dots, X_n$ be a random sample from $N(\mu, \mu^2)$, show that $(\sum X_i, \sum X_i^2)$ is sufficient for μ but not complete.
29. Based on a random sample of size n from Uniform(0, θ) obtain an unbiased confidence interval for θ with confidence level $1 - \alpha$.

30. Given three independent random variables Y_1, Y_2 and Y_3 having common variance σ^2 such that $E(Y_i) = \beta_0 + \beta_1 X_i, i = 1, 2, 3$ where $X_1 = -1, X_2 = 0$ and $X_3 = 1$. Find BLUEs of β_0 and β_1 .
31. Prove that the sequence $\{X_n\}$ of random variables converge to a random variable X a.s. iff :
- $$\lim_{n \rightarrow \infty} P\left(\bigcup_{m=n}^{\infty} |X_m - X| \geq \varepsilon\right) = 0 \text{ for every } \varepsilon > 0.$$
32. If $X \sim N_p(0, \Sigma)$, then state and prove a necessary and sufficient condition for $X'AX$ to have a Chi-square distribution.
33. Describe the analysis of 2^4 experiment in which the effects ABC and BCD are confounded with the blocks.
34. Define Horvitz-Thompson estimator of the population mean and derive an expression for its variance.
35. Let $\{X_n, n \geq 0\}$ be a branching process with $X_0 = 1$. Obtain the usual recurrence formulae for the p.g.f. of X_n .

(5 × 10 = 50 marks)