

Ph.D. ENTRANCE EXAMINATION, APRIL 2022
MATHEMATICS

Time : Two Hours

Maximum : 100 Marks

Part A

Answer all questions.
Each question carries 2 marks.

1. Consider the following statements :

- (I) Jan is well or Jan is still recovering.
- (II) If Jan is still recovering, then Jan is not well.
- (III) If Jan is well, then Jan is not still recovering.

Then which among the following are equivalent.

- (a) I and II.
 - (b) I and III.
 - (c) II and III.
 - (d) None.
2. Which statement is incorrect if X and Y are the two non-empty relations on a set S :
- (a) If X and Y are transitive, then the intersection of X and Y is also transitive.
 - (b) If X and Y are reflexive, then the intersection of X and Y is also reflexive.
 - (c) If X and Y are symmetric, then the union of X and Y is not symmetric.
 - (d) If X and Y are transitive, then the union of X and Y is not transitive.
3. The Initial value Problem :

$$y' = 3y^{2/3}, y(2) = 0 \text{ has}$$

4. The inverse Laplace transform of $\frac{8}{3s^2 + 12} + \frac{3}{s^2 - 49}$ is :

- (a) Unique solution.
 - (b) Two solutions.
 - (c) No solution.
 - (d) Infinitely many solutions.
- (a) $\frac{4}{3}\cos(2t) + \frac{3}{7}\cosh(7t)$.
 - (b) $\frac{4}{3}\sin(t) + \frac{3}{7}\sinh(4t)$.
 - (c) $\frac{3}{4}\sin(3t) + \frac{3}{7}\sinh(3t)$.
 - (d) $\frac{4}{3}\sin(2t) + \frac{3}{7}\sinh(7t)$.

Turn over

5. The type of singularity of $f(z) = \frac{1 + \cos(z)}{(z - \pi)^2}$ at $z = \pi$ is :
- (a) Pole of order 2. (b) Pole of order 1.
(c) Removable singularity. (d) Essential singularity.
6. In $\mathbb{R}$ with the co-finite topology, consider the following sequences :
- (i) $\{1, 2, 3, \dots\}$.
(ii) $\{1, 1, 2, 1, 3, 1, \dots\}$.
(iii) $\{1, 2, 1, 2, 1, 2, \dots\}$.
- Then
- (a) All are convergent. (b) All are not convergent
(c) Only (i) convergent. (d) Only (iii) is not convergent.
7. Let W and W' be subspaces of a vector space V such that $W \cup W'$ is also a subspace of V . Then which of the following is not likely to be true ?
- (a) $W \cup W' = W$. (b) $W \cup W' = W'$.
(c) $W \cup W' = \phi$. (d) $W = W'$.
8. The nature of the PDE $x^2 u_{xx} - 2xy u_{xy} - 3y^2 u_{yy} + u_y = 0$ for all values of x, y is :
- (a) Parabolic. (b) Hyperbolic.
(c) Elliptic. (d) None of the above.
9. Which among the following pairs of functions is linearly dependent on the given intervals ?
- (a) $\sin 4x, \cos 4x$ $(-\infty, \infty)$. (b) $\cos 2x, \sin^2 x$ $(0, \infty)$.
(c) $x^3, x^2|x|$ $[-1, 1]$. (d) None of the above.
10. Among the following numerical methods, which one is having the highest rate of convergence :
- (a) Secant. (b) Muller.
(c) Regula-Falsi. (d) Newton-Raphson.

11. Let H be a Hilbert space and let $\{e_i\}$ be an orthonormal set in H , then :

$$\|x\|^2 = \sum |\langle x, e_i \rangle|^2.$$

This equality is known as :

- (a) Parseval's identity. (b) Apollonius identity.
 (c) Holder's inequality. (d) None of these.
12. How many double transpositions are there in permutation group S_4 as elements.
 (a) 3. (b) 4.
 (c) 12. (d) 8.
13. Given the two points $[a, f(a)], [b, f(b)]$, the linear Lagrange interpolating polynomial that passes through these two points is given by :

(a) $\frac{x-b}{a-b}f(a) + \frac{x-a}{a-b}f(b).$ (b) $\frac{x}{b-a}f(a) + \frac{x}{b-a}f(b).$
 (c) $f(a) + \frac{f(b)-f(a)}{b-a}(b-a).$ (d) $\frac{x-b}{a-b}f(a) + \frac{x-a}{b-a}f(b).$

14. If $f(x)$ continuous in $[a, b]$ and $f(a)f(b) < 0$, then the equation $f(x) = 0$.
 (a) Can have no solution in $[a, b]$.
 (b) Can have two solutions in $[a, b]$.
 (c) Can have five solutions in $[a, b]$.
 (d) Can have six solutions in $[a, b]$.

15. Taylor polynomial of order two at $x = 0$ for the function $\sqrt{1+x}$ is :

(a) $1 + \frac{1}{2}x - \frac{1}{8}x^2.$ (b) $1 + \frac{1}{2}x - \frac{1}{4}x^2.$
 (c) $2 + \frac{1}{4}x - \frac{1}{8}x^2.$ (d) $1 + \frac{1}{8}x - \frac{1}{2}x^2.$

16. Suppose that :

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^3}, g(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$$

then

- (a) Both series converge uniformly and absolutely on $\mathbb{R}$.
- (b) Both series converge uniformly but not absolutely on $\mathbb{R}$.
- (c) Both are differentiable on $\mathbb{R}$ and f is the derivative of g .
- (d) Both are continuous but not differentiable on $\mathbb{R}$.

17. Radius of convergence of the power series $x^2 + x^3 + x^5 + x^7 + x^{11} + \dots$ is :

- (a) 1.
- (b) 2.
- (c) 0.
- (d) Infinity.

18. In dual simplex method, ————— variables are not required.

- (a) Slack.
- (b) Surplus.
- (c) Original.
- (d) Artificial.

19. The intersection of finite number of closed half spaces in $\mathbb{R}^n$ is called :

- (a) Convex null.
- (b) Polyhedral convex set.
- (c) Convex cone.
- (d) Simplex.

20. The eigen functions of symmetric kernel, corresponding to different eigen values are :

- (a) Real.
- (b) Imaginary.
- (c) Orthogonal.
- (d) Equal.

21. Hahn decomposition is unique expect for :

- (a) Positive sets.
- (b) Negative sets.
- (c) Null sets.
- (d) Measurable sets.

22. In a Hausdorff space, every sequence has :

- (a) Many limits.
- (b) No limit.
- (c) At most one limit.
- (d) Exactly one limit.

23. The convolution of $f(t) = e^{-t}$ and $g(t) = \sin(t)$ is :

(a) $(f * g)(t) = \frac{1}{2} [e^{-t} + \sin(t) - \cos(t)]$.

(b) $(f * g)(t) = \frac{1}{4} [e^t + \sin(t) + \cos(t)]$.

(c) $(f * g)(t) = \frac{1}{2} [e^{-t} - \sin(t) - \cos(t)]$.

(d) $(f * g)(t) = \frac{1}{4} [e^{-t} - \sin(t) - \cos(t)]$.

24. The integral equation :

$$g(s) = \frac{1}{2} + s - \int_0^1 (s-t) g^2(t) dt$$

is

- (a) Homogeneous linear Fredholm integral equation of the second kind.
- (b) Homogeneous non linear Volterra integral equation of the second kind.
- (c) Homogeneous linear Volterra integral equation of the second kind.
- (d) Non Homogeneous non linear Fredholm integral equation of the second kind.

25. The k^{th} Betti number of Klein bottle is :

- (a) k .
- (b) 1 for $k = 0, 1$ and 0 for all other values of k .
- (c) 1 for all values of k .
- (d) 0 for $k = 0, 1$ and 1 for all other values of k .

(25 × 2 = 50 marks)

Part B

Answer any **ten** questions.
Each question carries 5 marks.

26. Determine the range and null space of the linear transformation

$$T: \mathbb{R}^3 \rightarrow \mathbb{R}^4 \text{ with } T(x, y, z) = (x - y + z, y - z, x, 2x - 5y + 5z).$$

27. If $\sum a_n, \sum b_n$ are absolutely convergent and define :

$$c_n = \sum_{r=0}^n a_r b_{n-r}$$

Then show that $\sum c_n$ is absolutely convergent and $\sum c_n = \sum a_n \cdot \sum b_n$.

28. Let f be a continuous mapping defined from a topological space E to a topological space F . If $\{p_n\}$ converges to p in E , then show that $\{f(p_n)\}$ converges to $f(p)$ in F .

29. Prove that $\text{lcm}\{a, b\} \times \text{gcd}\{a, b\} = |ab|$.

where : $\text{lcm}\{a, b\}$ denotes the lowest common multiple of a and b and $\text{gcd}\{a, b\}$ denotes the greatest common divisor of a and b .

30. Prove that if f is a function that is continuous on the interval $[a, b]$, then f is Riemann-Stieltjes integrable on $[a, b]$.

31. Prove that the implication is left distribution with respect to the disjunction.

32. Find the image of the line passing through origin under the Joukowski Transform $w(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$.

33. Let (x_n) be a sequence in a normed space X . Show that $x_n \xrightarrow{\text{weakly}} x$ implies $\liminf_{n \rightarrow \infty} \|x_n\| \geq \|x\|$.

34. Let R be a ring. Let M, N be simple (or irreducible) sub modules of R , and let $\varphi: M \rightarrow N$ be a homomorphism of R -modules. Prove that either $\varphi = 0$, or φ is an isomorphism.

35. If $\{K_n\}$ be a sequence of nonempty compact sets in $\mathbb{R}^n$ satisfying $K_1 \supseteq K_2 \supseteq K_3 \dots$, then show that $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$.

36. Prove that it is impossible to trisect an angle with a ruler and compass.
37. Prove that there cannot be a continuous onto function $f: \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Q}$.
38. Solve the congruence equation $230x = 1081 \pmod{12167}$.
39. Apply the Gram-Schmidt orthonormalization process to the following basis for $\mathbb{R}^3$.
 $B = \{1,1,0\}, (1,2,0), (0,1,2)$ and get the orthonormal basis.
40. Show that the continuous image of a Lindelöf space is Lindelöf.

(10 × 5 = 50 marks)