

Ph.D. ENTRANCE EXAMINATION, APRIL 2021

MATHEMATICS

Time : Two Hours

Maximum : 100 Marks

Part A

*Answer all questions.**Each question carries 2 marks.*

1. What are the contrapositive of the conditional statement "I come to class whenever there is going to be a test" ?

- a) "If I come to class, then there will be a test."
- b) "If I do not come to class, then there will not be a test."
- c) "If there is not going to be a test, then I don't come to class."
- d) "If there is going to be a test, then I don't come to class."

2. $\neg(A \vee q) \wedge (A \wedge q)$ is a :

- a) Tautology.
- b) Contradiction.
- c) Contingency.
- d) None of these.

3. General solution of the differential equation

$$y''' + 2y'' - y' - 2y = 0 \text{ is :}$$

- a) $y(x) = C_1 e^{-2x} + C_2 e^x + C_3 e^{-x}$.
- b) $y(x) = C_1 e^{2x} + C_2 e^{-x} + C_3 e^{-2x}$.
- c) $y(x) = C_1 e^{-3x} + C_2 e^{-x} + C_3 e^{-2x}$.
- d) $y(x) = C_1 e^{2x} + C_2 e^x + C_3 e^{-x}$.

4. The Wronskian of two solutions to the differential equation

$$t^4 y'' - t^3 y' - t^8 y \text{ is :}$$

- a) ct^2 .
- b) $ct^2 + bt$.
- c) $\frac{c}{t}$.
- d) $\frac{c}{t^2}$.

Turn over

5. Residue of $\frac{e^z}{z^3}$ at $z = 0$ is:

a) 0.

b) $\frac{1}{4}$.

c) $\frac{1}{8}$.

d) $\frac{1}{2}$.

6. The set $\{0, 1\}$ furnished with the topology $\{\emptyset, \{0\}, \{0, 1\}\}$ is:

a) T0 and T1.

b) T0 but not T1.

c) T1 and T2.

d) T0, T1 and T2.

7. The upper Riemann sum of $f(x) = 2x + 1$ with respect to the partition $P = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}$ of the interval $[0, 1]$, $U(P, f)$, is:

a) $\frac{9}{4}$.

b) $\frac{7}{4}$.

c) $\frac{11}{4}$.

d) $\frac{3}{4}$.

8. Under the mapping $w(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$, the image of interior of $|z| = 1$ is:

a) The line segment $[-1, 1]$.

b) Complement of line segment $[-1, 1]$.

c) Ellipses with foci $-1, 1$.

d) Hyperbolas with foci $-1, 1$.

9. If x and y are any *two* vectors in an inner product space X , then :

$$| \langle x, y \rangle | \leq \| x \| \cdot \| y \|.$$

The above inequality is known as :

- (a) Parallelogram law.
 - (b) Cauchy-Schwarz's inequality.
 - (c) Polarisation identity.
 - (d) Bessel's inequality.
10. With respect to the partial ordering $m \leq n$ whenever m divides n , defined on the set $\{2, 3, 4, 8\}$:
- a) 8 and 3 are maximal.
 - b) All are maximal.
 - c) 3 only is maximal.
 - d) 8 only is maximal.
11. Which among the following is not true with respect to the permutation group S_3 ?
- a) 3 elements of S_3 are odd.
 - b) Non-abelian.
 - c) Cyclic.
 - d) 3 elements of S_3 are even.
12. The radius of convergence of the power series :

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^{n(n+1)} \text{ is}$$

- a) 1.
 - b) 2.
 - c) $\frac{1}{2}$.
 - d) 0.
13. Which among the following is not true ?
- a) $\frac{1}{x}$ is not uniformly continuous on $(0, 1]$.
 - b) x^2 is not uniformly continuous on $[0, \infty)$.
 - c) $\sqrt{x}$ is uniformly continuous on $[0, \infty)$.
 - d) None of the above.

Turn over

14. The value of $\lim_{x \rightarrow 0} \frac{a^{mx} - b^{nx}}{\sin kx}$ is:

a) $\frac{1}{k} \ln \frac{a^m}{b^n}$.

b) $\frac{1}{k} \ln \left(\frac{a}{b} \right)^{mn}$.

c) $\frac{1}{k} \ln \left(\frac{a}{b} \right)^{m-n}$.

d) $k \ln \left(\frac{a}{b} \right)^{mn}$.

15. If $K \subseteq \mathbb{R}^n$, which among the following is not correct ?

- a) If every sequence in K has a convergent subsequence, then K must be bounded.
- b) If every sequence in K has a convergent subsequence whose limit is in K , then K must be closed.
- c) If K is compact then any infinite subset $A \subseteq K$ has a cluster point in K .
- d) None of the above.

16. Which among the following is not necessarily true ?

- a) Cancellation laws hold in commutative rings with unity.
- b) Non-zero Elements of a Field form a group under multiplication.
- c) Every field is isomorphic to its own field of quotients.
- d) The only ideals in a field F are 0 and F .

17. Newton-Gregory Forward interpolation formula can be used :

- a) Only for equally spaced intervals.
- b) Only for unequally spaced intervals.
- c) For both equally and unequally spaced intervals.
- d) None of the above.

18. Select the true one for a Boolean Algebra B :

- a) B is a finite, complemented, and distributive lattice.
- b) B is a finite but not complemented lattice.
- c) B is a finite, distributive but not complemented lattice.
- d) B is a non-distributive lattice.

19. Consider an undirected random graph of eight vertices. The probability that there is an edge between a pair of vertices is $\frac{1}{2}$. What is the expected number of unordered cycles of length three ?

- a) $\frac{1}{4}$.
- b) 6.
- c) 7.
- d) 8.

20. What are the last two (decimal) digits of 3^{4242} ?

- a) 29.
- b) 19.
- c) 09.
- d) 79.

21. The orthogonal matrix associated with a rotation of R^3 of θ degrees about the z-axis is :

a)
$$\begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b)
$$\begin{bmatrix} \cos \theta & -\sin \theta & 1 \\ \sin \theta & \cos \theta & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

c)
$$\begin{bmatrix} -\cos \theta & \sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

d)
$$\begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

22. The situation of at least one artificial variable is positive in the optimum iteration will not happen in :

- a) Dual simplex method.
- b) Big-M Method.
- c) Two-Phase Method.
- d) All of the above.

Turn over

23. The integral equation $g(s) = s - \frac{1}{6}s^3 + \int_0^s (s-t)g(t)dt$ is :
- a) Homogeneous linear Fredholm integral equation of the second kind.
 - b) Non-Homogeneous linear Volterra integral equation of the second kind.
 - c) Homogeneous linear Volterra integral equation of the second kind.
 - d) Non-Homogeneous non-linear Fredholm integral equation of the second kind.
24. Poincare polynomial of Klein Bottle is :
- a) 1
 - b) x .
 - c) $1+x$.
 - d) $1+x+x^2$.
25. The functions $\sinh z$ and $\cosh z$ are :
- a) Bounded and entire.
 - b) Bounded but not entire.
 - c) Unbounded and not entire.
 - d) Unbounded and entire.

(25 × 2 = 50 marks)

Part B

*Answer any ten questions.
Each question carries 5 marks.*

26. Find general solution of the differential equations $(1-x^2)y'' - 2xy' + 2y = 0$ given that one solution is $y = x$.
27. Prove that an infinite subset of a compact space has at least one limit point.
28. Prove that $2^n + 1$ has no prime factors of the form $8k + 7$.
29. Find the general solution of the PDE $(y-z)p + (z-x)q = x-y$.

30. Give example of a space which is first countable, but not second countable. Also provide justifications for your claim.

31. Suppose f is bounded on $[a, b]$ and continuous at $s \in (a, b)$. If $\alpha(x) = \chi_{(0, \infty)}(x - s)$, prove that

$$\int_a^b f(x) d\alpha(x) = f(s)$$

$$\text{where } \chi_{(0, \infty)}(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases}$$

32. Show that a Banach space X is reflexive if and only if its dual space X' is reflexive.

33. Prove there is no branch of the logarithm defined on $G = \mathbb{C} \setminus \{0\}$.

34. Solve the system of congruence equations :

$$2x \equiv 6 \pmod{14}$$

$$3x \equiv 9 \pmod{15}$$

$$5x \equiv 20 \pmod{60}.$$

35. Prove or Disprove : Every path connected space is connected.

36. Solve :

$$z(t) = \alpha \cos(\omega t) + \beta \int_0^t \sin(\omega(t-u)) z(u) du.$$

37. (i) Prove or disprove : Geodesics are having constant speed.

(ii) Find velocity and acceleration of the parametrized curve

$$\alpha(t) = (\cos t, \sin t, 2 \cos t, 2 \sin t).$$

38. Show that the torus less one point, the Klein bottle less one point and $\mathbb{R}^2$ less two points are each homotopy equivalent to the space obtained by gluing two disjoint circles together at a point.

Turn over

39. Find the maximum and minimum values of the function $f(x, y) = 3x + y$ subject to the constraint $x^2 + y^2 = 10$.

40. Find the Hermite interpolating polynomial $f(x)$ satisfying

$$f(0) = 2, f'(0) = 1, f(1) = 4, f'(1) = -1, f(3) = 5, f'(3) = -2.$$

(10 × 5 = 50 marks)